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Quantum mechanics and field theory on multiply connected and on homogeneous spaces

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Abstract. The basic framework for discussing quantum mechanics on multiply connected spaces is presented using the covering space concept. The theorem of Laidlaw and DeWitt is rederived and extended to the case of field theory. It is pointed out that chiral dynamics is similar to Skyrme's nonlinear theory and forms another example of Finkelstein's kink idea. The possible existence of 'π geons' is raised, and the fact that the pion manifold may be any one of the Clifford–Klein constant curvature space-forms, rather than just the whole three-sphere, is suggested. The related formalism for quantum mechanics on homogeneous spaces is given in general terms.

1. Introduction

Recently Laidlaw and DeWitt (1971) have discussed quantum mechanics on multiply connected spaces using the Feynman functional formulation. The propagator is given as the sum of 'partial amplitudes', each corresponding to a distinct homotopy class of Feynman paths. The coefficients in this sum are shown to be a unitary, one dimensional representation of the fundamental group of the space (see also Schulman 1971). In this paper we should like to give what we consider to be a neater presentation of this result, to extend it to the case of field theory and also to discuss the, in some ways related, problem of quantum mechanics on a homogeneous space. In previous papers (Dowker 1970, 1971) we have considered quantum mechanics on a semisimple Lie group and we would like to extend our results in some way to the more general class of homogeneous spaces.

2. Multiply connected spaces

The most sensible way of dealing with a multiply connected space $\mathscr{M}$ is through its covering spaces, in particular its universal covering space $\tilde{\mathscr{M}}$. Most textbooks on topology discuss this concept at some length and we shall only mention the works by Steenrod (1951), Whitehead (1966) and Wolf (1967) as being especially relevant. The textbook by Hu (1959) is also an important source of information.

In essence, $\tilde{\mathscr{M}}$ is defined by a map $\tilde{\mathscr{M}} \to \mathscr{M} = \tilde{\mathscr{M}}/\Gamma$ where Γ is a properly discontinuous, discrete group of isometries of $\tilde{\mathscr{M}}$, without fixed points. $\tilde{\mathscr{M}}$ is simply connected and the fundamental path group of $\mathscr{M}$, $\Pi_1(\mathscr{M})$, is isomorphic to Γ.

In pictorial terms, $\mathscr{M}$ is obtained from $\tilde{\mathscr{M}}$ by identifying points equivalent under Γ. Since Γ is discrete $\mathscr{M}$ and $\tilde{\mathscr{M}}$ have the same dimension and are locally isometric if we assume, as we now do, that they are Riemannian spaces.

Let us denote a general point of $\tilde{\mathcal{M}}$ by $\tilde{q}$ and of $\mathcal{M}$ by q. The meaning of the quotient space $\mathcal{M} = \tilde{\mathcal{M}}/\Gamma$ is that each point q of $\mathcal{M}$ corresponds to the n different points $\tilde{q}\gamma$ of $\tilde{\mathcal{M}}$ where γ ranges over the n elements of Γ. (We write the effect of Γ on $\tilde{\mathcal{M}}$ to the right, $\tilde{q} \rightarrow \tilde{q}\gamma$.) $\tilde{\mathcal{M}}$ is thus divided into subsets of a finite number of points, or 'fibres', as these subsets are called, one fibre corresponding to one point of $\mathcal{M}$. (In topological terminology $\tilde{\mathcal{M}}$ is a bundle, that is, a fibred space, and Γ is the group of the bundle. Together with the base space $\mathcal{M}$ and the mapping $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$, Γ and $\mathcal{M}$ make up a fibre bundle (eg Steenrod 1951, §§ 1.6, 1.7).)

It is convenient to choose a fixed 'base' point $\tilde{q}_0$ in each of the fibres. The set $\bar{F}$ of all these base points is clearly isomorphic to $\mathcal{M}$ and since every point of $\tilde{\mathcal{M}}$ can be represented in the form

$$\tilde{q} = \tilde{q}_0\gamma. \tag{1}$$

F, the complement of $\bar{F}$ in $\tilde{\mathcal{M}}$, is a so called fundamental domain in $\tilde{\mathcal{M}}$ relative to Γ (eg Gelfand *et al* 1969, p 5).

3. Quantum mechanics

Since $\tilde{\mathcal{M}}$ is simply connected a wavefunction on it, $\tilde{\psi}(\tilde{q})$, will be single valued. We can now define (cf Finkelstein 1966) a multivalued wavefunction on $\mathcal{M}$, $\psi(q)$, by saying that $\psi(q)$ has the values taken by $\tilde{\psi}(\tilde{q}_0\gamma)$ where $\tilde{q}_0$ corresponds to q and γ ranges over all Γ. Choosing the particular correspondence $\tilde{q}_0 \rightarrow q$, that is $\gamma = 1$, gives us a single valued function on $\mathcal{M}$, $\psi(q) = \tilde{\psi}(\tilde{q}_0)$. A further requirement is that $\psi(q)$ should be continuous. We assume that $\tilde{\psi}(\tilde{q})$ is so. Thus, as q'' tends to q' we must have that $\tilde{q}_0''$ tends to $\tilde{q}_0'$, so that if one $\tilde{q}_0$ is selected the whole set $\bar{F}$ will be determined. It is clear that there will be as many $\bar{F}$ as there are elements of Γ, each obtained from any other by a translation belonging to Γ. The corresponding wavefunctions will be the branches of our multi-valued wavefunction on $\mathcal{M}$.

For physical reasons we shall require that these branches be equivalent. This means that they have the same modulus, that is

$$\tilde{\psi}(\tilde{q}_0\gamma) = a(\gamma)\tilde{\psi}(\tilde{q}_0) \tag{2}$$

where

$$|a(\gamma)| = 1.$$

It is easy to show from (2) that $a(\gamma)$ must be a representation of Γ, that is

$$a(\gamma_1\gamma_2) = a(\gamma_2)a(\gamma_1) \tag{3}$$

up to a constant phase.

In order to make contact with the work of Laidlaw and DeWitt we now introduce the propagators. Thus, on $\tilde{\mathcal{M}}$, we have

$$\tilde{\psi}(\tilde{q}'', t'') = \int_{\tilde{\mathcal{M}}} \tilde{K}(\tilde{q}''t''|\tilde{q}'t')\tilde{\psi}(\tilde{q}', t')\,\mathrm{d}\tilde{q}'.$$

The integration over $\tilde{\mathcal{M}}$ is split up into an integration over an $\bar{F}$ and a sum over Γ. If (1) and (2) are used this yields

$$\tilde{\psi}(\tilde{q}_0'') = \int_{\bar{F}} \left(\sum_{\gamma'} \tilde{K}(\tilde{q}_0''|\tilde{q}_0'\gamma')a(\gamma') \right) \tilde{\psi}(\tilde{q}_0')\,\mathrm{d}\tilde{q}_0' \tag{4}$$

where we have dropped the time variable. It is now only necessary to use the isomorphism of $\bar{F}$ and $\mathscr{M}$ and to replace $\tilde{q}_0''$ and $\tilde{q}_0$ by their corresponding points, q'' and q', in $\mathscr{M}$ in order to obtain the propagation law on $\mathscr{M}$

$$\psi(q'') = \int_{\mathscr{M}} K(q''|q')\psi(q')\,\mathrm{d}q' \tag{5}$$

with

$$K(q''|q') = \sum_{\gamma'} \tilde{K}(\tilde{q}_0''|\tilde{q}_0'\gamma')a(\gamma'). \tag{5'}$$

We emphasize that here $\psi(q) = \tilde{\psi}(\tilde{q}_0)$ is a single valued continuous function on $\mathscr{M}$.

Equation (5), with equation (3), constitutes the result of Laidlaw and DeWitt previously referred to. This is because $\Pi_1(\mathscr{M}) \simeq \Gamma$ and part of the paper of Laidlaw and DeWitt is essentially a proof of this known result of homotopy theory (eg Steenrod 1951, § 14).

For consistency we want equation (2) to be valid for all times, which means that it must be propagated by equation (4). For this it is necessary that Γ be an invariance group of the quantum system, which implies that

$$\tilde{K}(\tilde{q}''\gamma|\tilde{q}'\gamma) = \tilde{K}(\tilde{q}''|\tilde{q}')$$

or equivalently

$$\tilde{K}(\tilde{q}''\gamma|\tilde{q}') = \tilde{K}(\tilde{q}''|\tilde{q}'\gamma^{-1}). \tag{6}$$

Using this condition, equation (3) and the invariance of the sum over Γ under a constant translation by γ it is easily shown that

$$\tilde{\psi}(\tilde{q}_0''\gamma, t) = a(\gamma)\tilde{\psi}(\tilde{q}_0'', t)$$

as required.

Since Γ is a group of isometries of $\tilde{\mathscr{M}}$ it will be sufficient for the invariance of the quantum system under Γ if the equation of motion, that is Schrödinger's equation, on $\tilde{\mathscr{M}}$ is covariant under coordinate transformations of $\tilde{\mathscr{M}}$, for example

$$\mathrm{i}\dot{\tilde{\psi}}(\tilde{q}, t) = -\tfrac{1}{2}\tilde{\Delta}_2\tilde{\psi}(\tilde{q}, t) \tag{7}$$

where $\tilde{\Delta}_2$ is the Laplace–Beltrami operator on $\mathscr{M}$. Because $\mathscr{M}$ is locally isometric to $\tilde{\mathscr{M}}$, $\psi(q)$ will satisfy the same Schrödinger equation as $\tilde{\psi}(\tilde{q})$, namely

$$\mathrm{i}\dot{\psi}(q, t) = -\tfrac{1}{2}\Delta_2\psi(q, t) \tag{8}$$

where

$$\tilde{\Delta}_2(q)|_{\tilde{q}=q} = \Delta_2(q).$$

Equation (5′) can be interpreted in terms of images (cf Schulman 1971). The total amplitude in $\mathscr{M}$ is obtained by summing the (partial) amplitudes in $\tilde{\mathscr{M}}$ from each of the (pre-)images $\tilde{q}_0\gamma'$ of the initial point q' to a fixed image $\tilde{q}_0''$ of the final point q''. Using (6) we can equivalently sum over images of the final point, for a fixed initial point image.

4. Quotient and homogeneous spaces

The space $\mathscr{M} = \tilde{\mathscr{M}}/\Gamma$ is a particular example of a quotient space and we can generalize the theory of the preceding section to the more general case $\mathscr{M} = \tilde{\mathscr{M}}/H$ where H is a continuous Lie group of isometries of $\tilde{\mathcal{M}}$. We need consider only the case where H consists of one connected part. (The more general case of several disconnected parts will follow from this special case and the theory of § 3, the fundamental group of $\mathcal{M}$ being just the group of the disconnected parts of H.)

The fibres which make up $\tilde{\mathcal{M}}$ are now continuous and $\mathcal{M}$ and $\tilde{\mathcal{M}}$ have different dimensions. A function on $\mathcal{M}$ can be considered to be a function on $\tilde{\mathcal{M}}$ constant on these fibres, that is, there is a mapping

$$\tilde{\psi}(\tilde{q}_0 h) = \tilde{\psi}(\tilde{q}_0) \rightarrow \psi(q) = \tilde{\psi}(\tilde{q}_0) \tag{9}$$

where again $\tilde{q}_0$ is a base point in a typical fibre such that any point in $\tilde{\mathcal{M}}$ can be represented by

$$\tilde{q} = \tilde{q}_0 h.$$

The propagator on $\mathcal{M}$ is now obtained in exactly the same way as in § 3 except that summation over Γ is replaced by integration over H. Omitting the steps we find

$$K(q''|q') = \int_H \tilde{K}(\tilde{q}''_0|\tilde{q}'_0 h)\, \mathrm{d}h \tag{10}$$

which we can possibly think of in terms of continuous images. The knowledgeable reader will recognize this as similar to the discussion of Wigner (1954) in his interesting paper on multiple scattering.

If the mapping (9) is to be propagated in time we again need that the dynamics should be invariant under H which means, for example, that the h in (10) can be switched from the last argument of $\tilde{K}$ to the first, where it becomes h^{-1}. This invariance will follow if $\tilde{\psi}$ and ψ satisfy the Schrödinger equations (7) and (8) but where now Δ_2 is the restriction of $\tilde{\Delta}_2$ to its action on functions satisfying (9).

Formally, it would be possible to include a factor $a(h)$ in (9), just as in (2), and *then* enquire as to its meaning. We just mention this possibility here and point out that if $a(h)$ is to be a nontrivial representation of H this must be an abelian group and, further, $\psi(q)$ will no longer satisfy equation (8). We should also like to draw attention to the fact that this modification bears some resemblance to the theory of induced representations (eg Mackey 1968, Vilenkin 1968, Gelfand *et al* 1969).

An especially important case is when $\tilde{\mathcal{M}}$ is the manifold of a Lie group, that is, when $\mathcal{M}$ is a homogeneous space $\mathcal{M} = G/H$. It may be that G is not simply connected. However we can always introduce *its* universal covering space, if desired, and employ (10) and (5′) in turn. There is, therefore, no loss in assuming G simply connected, and we can further assume that it is semisimple. Equation (5′), if $H = \Gamma$ is discrete, or equation (10), if H is continuous, will determine the propagator on G/H as an 'image sum' of propagators on a semisimple group. These we have already calculated (Dowker 1970, 1971) and have shown that the quasiclassical expression is exact up to a nonsignificant phase factor. Thus we can also say that the quasiclassical result is likewise exact on G/H, although if H is continuous this statement is not to be taken as literally true. For example, the propagator on a two dimensional sphere S^2 is not given by the quasiclassical expression calculated on the basis of S^2 kinematics but it *is* a (continuous) sum of quasiclassical propagators on SO(3). In the next section we discuss this standard example.

5. Example of SU(2)

The case when $G = SU(2)$, that is $\tilde{\mathcal{M}} = S^3$ the three dimensional sphere, has been treated at length by Schulman (1968) (see also DeWitt 1969) and corresponds classically to the spherical top. We shall here rederive Schulman's result that the propagator on the quotient space

$$\mathcal{M} = \frac{S^3}{Z_2} = P^3 = \frac{SU(2)}{Z_2} = SO(3)$$

allows a separation of integral and half integral spin.

The propagator corresponding to the Schrödinger equation (7) is given in terms of the SU(2) representation matrices $\mathscr{D}^l_{mn}(\tilde{q})$, $l = 1, 2, \ldots$ by

$$\tilde{K}(\tilde{q}''|\tilde{q}') = \sum_l A_l \mathscr{D}^l_{mn}(\tilde{q}'')\bar{\mathscr{D}}^l_{mn}(\tilde{q}') \qquad (l = 2j+1) \tag{11}$$

where the A_l are (known) coefficients depending on the time difference $t''-t'$.

$\tilde{K}$ is a function of $\tilde{q}''\tilde{q}'^{-1}$ only as it should be from the invariance of the theory under the (largest) group of isometries of S^3, $SU(2) \otimes SU(2)$.

There are two unitary, one dimensional representations of Z_2, namely $(1, 1)$ and $(1, -1)$ and hence, from (5′), two propagators on SO(3), K_+, K_- given by

$$K_\pm(q''|q') = \tilde{K}(\tilde{q}''_0|\tilde{q}'_0) \pm K(\tilde{q}''_0|-\tilde{q}'_0). \tag{12}$$

If we substitute (11) into (12) and use

$$\mathscr{D}^l(-\tilde{q}) = (-1)^{l+1}\mathscr{D}^l(\tilde{q})$$

we find

$$K_\pm(q''|q') = \sum_l A_l\{1 \mp (-1)^l\}\mathscr{D}^l_{mn}(\tilde{q}''_0)\bar{\mathscr{D}}^l_{mn}(\tilde{q}'_0)$$

which shows that K_+ contains only odd l, that is integral spins, while K_- has only half integral spins, as discussed by Schulman (1968)†.

There seems to be no reason why we should restrict ourselves to $\Gamma = Z_2$. Indeed, Seifert and Threlfall (1930), in a piece of classical mathematics, determined all spaces locally isometric to S^3 (with some reasonable restrictions such as completeness). There are, in fact, an infinite number of them. For example we can have S^3/Z_m, $m > 2$. We refer to Wolf (1967) for a modern treatment of this 'Clifford–Klein space-form problem'.

It is important to remember that the only space-forms S^3/Γ for which the largest group of isometries is $SU(2) \otimes SU(2)$ are S^3 itself and P^3. In general the largest, complete group of isometries of $\tilde{\mathcal{M}}/\Gamma$ is smaller than that of $\tilde{\mathcal{M}}$ although this latter still exists as a *local* group of isometries of $\tilde{\mathcal{M}}/\Gamma$.

Exactly what physics, if any, is to be attached to these spherical space-forms we do not know and we simply indicate their existence at this point.

A similar comment is valid for the construction of propagators corresponding to a different choice for G. For example we might make G SU(3) and calculate the propagator on $SU(3)/Z_3$, Z_3 being the centre of SU(3), by analogy with the SU(2) case, but the lack of a relevant mechanical or physical system seems to make this, and the like, calculations somewhat academic.

† This conclusion also follows directly from the expansion of $\tilde{\psi}(\tilde{q})$ in the $\mathscr{D}^l(\tilde{q})$.

There is one scheme, however, for which these considerations may be significant. We refer to the theory of chiral dynamics. This is a field theory and so we now turn to the modifications necessary in such a case and come back to chiral theory later.

6. Field theory

Instead of $q(t)$ we now have $q(\boldsymbol{x}, t)$. The formalism employed depends on whether we keep $\boldsymbol{x}$ with t or whether we treat $\boldsymbol{x}$ as a coordinate of q. In the latter case the appropriate place to discuss the dynamics is the functional space M whose points determine a function $q(\boldsymbol{x})$. M will be the topological product of all the $\mathcal{M}$ spaces, one attached to each space point $\boldsymbol{x}$

$$M = \prod_{\boldsymbol{x}} \mathcal{M}_{\boldsymbol{x}}.$$

M takes on the aspect of a mapping space (eg Hu 1959, p 73), $\mathcal{M}^X$, of the space X of $\boldsymbol{x}$ (here, three dimensional euclidean space) into $\mathcal{M}$. There are further topological complications for such a space. It may consist of disconnected pieces, if a boundary condition on $q(\boldsymbol{x}, t)$ is imposed at spatial infinity say, even if $\mathcal{M}$ is connected (by connected we mean path connected). A field distribution corresponding to a point in one of these pieces cannot develop into a field described by a point in a different part. Such fields would not be homotopic and the theory 'admits kinks', in the terminology of Finkelstein (1966) (see also Finkelstein and Rubinstein 1968, Finkelstein and Misner 1959, Williams 1970, 1971).

The group formed by the disconnected parts of M is the third homotopy group, $\Pi_3(\mathcal{M})$, of $\mathcal{M}$. In the present work we are not so much interested in the question of kinks as in the propagator within one of the connected components M_i of M. The formalism is exactly as in § 2 (see also Finkelstein and Rubinstein 1968, II). We introduce the universal covering space $\tilde{M}_i$ of M_i by

$$M_i = \frac{\tilde{M}_i}{\Delta_i}$$

where Δ_i is a discrete group of isometries of $\tilde{M}_i$, being in fact the fundamental group of M_i, $\Pi_1(M_i)$. Homotopy theory (the above references give the details) shows that

$$\Pi_1(M_i) \simeq \Pi_4(\mathcal{M})$$

for all i. $\Pi_4(\mathcal{M})$ is the fourth homotopy group of $\mathcal{M}$.

Just as before we introduce the state functionals $\tilde{\Psi}_i[\tilde{q}, t]$, $\Psi_i[q, t]$, on $\tilde{M}_i$ and M_i respectively, and write

$$\tilde{\Psi}_i[\tilde{q}_0\delta] = a_i(\delta)\tilde{\Psi}_i[\tilde{q}_0]$$

where $\tilde{\Psi}_i[\tilde{q}_0]$ is a single valued, continuous functional on the complement $\bar{F}_i$ of the fundamental domain F_i in $\tilde{M}_i$ with respect to Δ_i. Thus $\Psi_i[q] = \tilde{\Psi}_i[\tilde{q}_0\delta]$ defines the branches of the state functional on M_i, and again we require a_i to have unit modulus.

The functional propagator on the connected component M_i of M is

$$K_i[q''|q'] = \sum_{\delta'} \tilde{K}_i[\tilde{q}''_0|\tilde{q}'_0\delta']a_i(\delta') \tag{13}$$

where $a_i(\delta)$ is a unitary, one dimensional representation of $\Pi_4(\mathcal{M})$. This is the field-theoretic generalization of the result of Laidlaw and DeWitt (1971). It is, in fact, just what one would have expected from the Feynman formulation. In field theory the 'paths' become $(n+1)$ dimensional manifolds, if n is the dimension of the space X. The group that catalogues the homotopy classes of such manifolds is $\Pi_{n+1}(\mathcal{M})$.

An important theorem (eg Steenrod 1951, § 17.6, Hu 1959, proposition 7.1) states that $\Pi_n(\mathcal{M})$ is isomorphic to $\Pi_n(\tilde{\mathcal{M}})$ if n is *greater than two*. Thus, for field theory, the number of distinct functional propagators for $\mathcal{M}$ is the same as that for its universal covering space $\tilde{\mathcal{M}}$ in sharp contrast to the situation in the quantum theory of discrete systems discussed in § 3.

7. A possible application: chiral dynamics

In chiral SU(2) theory the manifold of the pion field variables $\phi^\alpha(\boldsymbol{x}, t)$ is taken to be the group manifold of SU(2), that is S^3, (see eg Meetz 1969, Isham 1969). Accordingly, the general theory of the preceding section is applicable and it will be of interest to pursue its consequences a little further for this case of physical interest. There do not seem to be many systems of real physical import whose configuration spaces are topologically interesting.

The general formalism for $\tilde{\mathcal{M}} = S^3$ has, in fact, been discussed by Finkelstein (1966) and by Williams (1970, 1971) and the main object of the present section is to point out its relevance for chiral dynamics. Up to the present this has not been recognized. Unfortunately any *dynamical* calculations are rather formidable and we cannot present any concrete conclusions at the present time. However we do wish to make two distinct points of qualitative, but ultimately of quantitative, significance.

The first point concerns the actual manifold of the pion variables. Do we need to take this to be either S^3 or, possibly, P^3 (as discussed by Isham 1969)? Up to now most, if not all, chiral theory calculations use a perturbation approach in order to extract physical quantities such as the pion–pion scattering amplitudes. The pion field is expanded about the value zero and so only local properties of the pion manifold are encountered. Thus, so far as present day tests of the theory go, $\mathcal{M}$ could be any of the space-forms S^3/Γ, locally isometric to S^3, mentioned in § 5†. We would like to raise, but not to answer, the question whether these could be distinguished experimentally. Roughly speaking, the pion field might become so large that the topological properties of $\mathcal{M}$ become important. It seems unlikely that we could test this directly. If we crudely extrapolate the Yukawa potential down to small distances just to get a rough idea, the pion field attains a size comparable with the circumference (about 2·8 fm) of S^3 at about 0·1 fm. Which is somewhat small.

The second, and potentially more important, point arises when we come to consider the pion field functional propagator, which will be given by equation (13), according to the general theory of § 6.

Since $\Pi_4(S^3) = Z_2$ there will be two sorts of propagator exactly as in § 5. In the quantum mechanical case we seized on this circumstance as providing evidence for a classical theory leading to quantum half-odd spins. Is there a correspondingly physical conclusion for fields? We might predict the existence of some quantized semilocal structure of the pion field with half-odd isotopic spin (the nucleon?). We could term this object a 'π geon' (see Finkelstein and Misner 1959). It need not be emphasized that these considerations are highly speculative.

† See also Mickelsson and Niederle (1970).

A curious circumstance arises if we try to extend these ideas to chiral SU(3), that is, if we include the K and η mesons as fundamental coordinates along with the pions. For, while $\Pi_3(SU(3)) = Z$ showing the existence of kinks, $\Pi_4(SU(3))$ is trivial and the theory does not admit the SU(3) analogue of half-odd spin. This situation has been analysed further and the results are contained in our paper 'Are Quarks Possible?', submitted for publication. For this reason no more will be said here.

Finally we should like to mention the investigations of Skyrme (eg Skyrme 1958, 1959, 1961) into a nonlinear field theory which bears, in some points, a remarkable resemblance to chiral theory and which forms an early example of kink theory. Skyrme's analysis is important because he is principally concerned with dynamical calculations.

Acknowledgments

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